

Enhanced Time-Series Water Wave Model through Refinement of Convective Acceleration and Driving Force in the Velocity Equation

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Abstract— *This study investigates the simplest time series type of the Boussinesq equation, namely the Airy long wave model. The primary limitation of this model is its applicability only to short wave conditions with very small amplitudes, which are significantly smaller than the maximum wave amplitude permitted for a given wave period. The source of this limitation is identified as the treatment of convective acceleration in the water particle velocity equation. Model refinement is introduced by interpreting the convective acceleration as a hydrodynamic force whose positive direction extends from regions of higher kinetic energy toward regions of lower kinetic energy. This interpretation requires assigning a negative sign to the convective acceleration term. In addition, the hydrostatic force commonly employed corresponds to the surface hydrostatic force, whereas the velocity equation utilizes depth averaged velocity. To ensure consistency, this force must therefore be reformulated as the depth averaged hydrostatic force.*

I. INTRODUCTION

Time series models and the velocity potential theory are among the most widely applied approaches for analyzing wave hydrodynamics in water bodies. Both approaches have their respective strengths and limitations. The velocity potential model is unable to simulate rip currents, whereas time series models are capable of representing such phenomena.

The most extensively developed and applied time series model is derived from the Boussinesq equation formulated in 1871. This model has been extensively researched and refined by numerous scholars, including Peregrine (1967), Hamm, Madsen, and Peregrine (1993), Nwogu (1993), Dingemans (1997), Johnson (1997), Madsen and Schaffer (1998), and Kirby (2003), among others.

The Airy long-wave equation was initially introduced to describe very long waves, particularly tidal waves (Dean, 1991). Subsequent research on the Boussinesq equation has retained the same fundamental structure as the Airy long-wave equation. Therefore, the Airy long-wave equation is often regarded as a form of the Boussinesq equation.

The present research investigates the application of the Airy long-wave model to short waves. The findings indicate that for short waves, the model can only be applied when the wave amplitude is extremely small, considerably below the maximum wave amplitude associated with a given wave period. To overcome this limitation, later modifications were introduced.

II. BASIC EQUATIONS OF HYDRODYNAMICS

There are two fundamental equations in hydrodynamics; the continuity equation and the conservation of momentum equation.

a. Continuity Equation

The continuity equation is derived from the principle of conservation of mass. In the two-dimensional plane (x, z) as follows.

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \dots (1)$$

where the x -axis represents the horizontal direction and the z -axis represents the vertical direction. The variable u denotes the velocity of a water particle in the horizontal x direction, while w denotes the velocity of a water particle in the vertical z -direction. This formulation is based on the assumption that the horizontal velocity varies only along the horizontal x -axis and the vertical velocity varies only along the vertical z -axis.

b. Euler's Momentum Conservation Equation.

Euler's momentum conservation equation for flow in the two-dimensional (x, z) plane consists of two separate formulations: the equation governing particle velocity along the horizontal axis and the equation governing particle velocity along the vertical axis. These equations are derived under the assumption that the horizontal velocity varies along both the horizontal x -axis and the vertical z -axis, while the vertical velocity undergoes corresponding variations. This condition differs from that applied in the formulation of the continuity equation.

The equation for particle velocity along the horizontal axis is expressed as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad \dots (2)$$

The equation for particle velocity along the vertical axis is expressed as,

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad \dots (3)$$

Where ρ denotes the water density, p is the pressure on the fluid particle, and g is the gravity.

III. RESEARCH ON THE AIRY LONG-WAVE EQUATION

This section examines the convective acceleration term in the Airy long-wave model. The model is formulated using

equations (1), (2), and (3). For brevity, the full derivation is not presented here, as it can be found in Dean (1991). The model consists of two principal equations.

The first equation represents the water surface elevation. It is obtained by integrating equation (1) with respect to the vertical z -axis and applying both the kinematic free surface boundary condition and the kinematic bottom boundary condition,

$$\frac{\partial \eta}{\partial t} + \frac{\partial UH}{\partial x} = 0 \quad \dots (4)$$

The second equation describes the horizontal motion of water particles. It is derived by integrating equations (2) and (3) with respect to the vertical z -axis:

$$\frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial UU}{\partial x} = -g \frac{\partial \eta}{\partial x} \quad \dots (5)$$

In equation (5), the second term on the left-hand side represents the convective acceleration, while the right-hand side corresponds to the hydrostatic force. In equations (4) and (5), U denotes the depth-averaged horizontal velocity of water particle, η is the water surface elevation relative to the still-water level and H is the total water depth.

$$H = h + \eta$$

Where h is the water depth measured relative to the still-water level, see Fig.1.

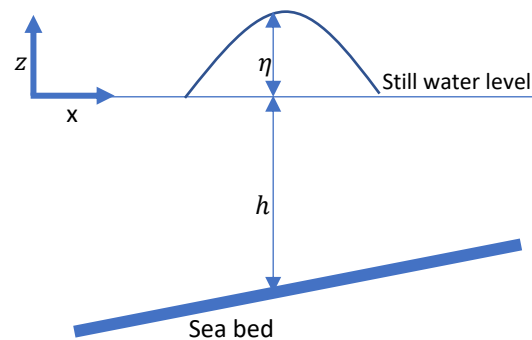


Fig.1: Water surface elevation and water depth

3.1. Numerical Method.

To solve the time series water wave equation, the finite difference method is employed for spatial discretization, while time differentials are addressed using a predictor–corrector approach. In the predictor stage, the finite difference method is applied with a central-difference scheme, whereas the corrector stage utilizes an integration method based on Newton–Cote numerical integration. A detailed description of this predictor–corrector method is provided in Hutahaean (2024a).

a. Calculation of the Time-Step Value δt .

As noted above, the predictor stage employs the finite difference method with a central-difference scheme. In this approach, the finite difference equation is obtained by truncating the Taylor series expansion to the second-order term. It is assumed that for sufficiently small values of δt , the third-order and higher-order terms are negligible and may therefore be disregarded.

Taylor series for a function of time (t) is

$$f(t + \delta t) = f(t) + \delta t \frac{df}{dt} + \frac{\delta t^2}{2} \frac{d^2f}{dt^2} + \frac{\delta t^3}{6} \frac{d^3f}{dt^3} + \frac{\delta t^4}{24} \frac{d^4f}{dt^4} + \dots$$

This equation can be truncated to order 1 only if,

$$\left| \frac{\frac{\delta t^2}{2} \frac{d^2f}{dt^2} + \frac{\delta t^3}{6} \frac{d^3f}{dt^3} + \frac{\delta t^4}{24} \frac{d^4f}{dt^4} + \dots}{\delta t \frac{df}{dt}} \right| < \varepsilon \quad \dots (6)$$

Where ε is an extremely small number.

At the smallest δt , the following equations apply

$$\left| \frac{\delta t^4}{24} \frac{d^4f}{dt^4} + \frac{\delta t^5}{120} \frac{d^5f}{dt^5} + \frac{\delta t^6}{720} \frac{d^6f}{dt^6} \right| \ll \left| \frac{\delta t^2}{2} \frac{d^2f}{dt^2} + \frac{\delta t^3}{6} \frac{d^3f}{dt^3} \right|$$

Thus (6) becomes,

$$\left| \frac{\frac{\delta t^2}{2} \frac{d^2f}{dt^2} + \frac{\delta t^3}{6} \frac{d^3f}{dt^3}}{\delta t \frac{df}{dt}} \right| < \varepsilon$$

Or

$$\left| \frac{\frac{\delta t^2}{2} \frac{d^2f}{dt^2} + \frac{\delta t^3}{6} \frac{d^3f}{dt^3}}{\frac{df}{dt}} \right| \leq \varepsilon \quad \dots (7)$$

Considering a time dependent function t , $f(t) = \cos \sigma t$, substituted to (7)

$$\left| \frac{-\frac{\delta t}{2} \sigma^2 \cos \sigma t + \frac{\delta t^2}{6} \sigma^3 \sin \sigma t}{-\sigma \sin \sigma t} \right| \leq \varepsilon$$

This equation applies within the characteristic point where $\sin \sigma t = \cos \sigma t$,

$$\left| \frac{\delta t}{2} \sigma - \frac{\delta t^2}{6} \sigma^2 \right| \leq \varepsilon$$

Assuming the expression inside the absolute value is positive and setting it equal gives the quadratic condition,

$$-\frac{\delta t^2}{6} \sigma^2 + \frac{\delta t}{2} \sigma - \varepsilon = 0 \quad \dots (8)$$

With input wave period T , where $\sigma = \frac{2\pi}{T}$, and determining $\varepsilon = 0.005$, thus δt follow from equation (8). There are two values of δt , where the smaller one is used.

The value δt obtained from equation (8) ensures that the neglected terms in the Taylor series are bounded by ε . The value δt where Taylor Series can be truncated into one order, the truncation also applies to order 2.

b. Grid-Size Calculation (δx).

Grid-size δx was calculated out using the modified Courant criterion (1928) as follows.

$$\delta x = 3.1 \frac{\sigma}{k} \delta t \quad \dots (9)$$

In the original Courant criterion, a coefficient of 3.0 is used, where k denotes the wave number. Considering that the wavelength in the Airy long-wave equation is very large, approaching the wavelength predicted by linear wave theory, the deep-water wave number can be approximated using the dispersion relation from linear wave theory, as follows.

$$k_0 = \frac{\sigma^2}{g}$$

$$\text{Deep water wavelength } L_0 = \frac{2\pi}{k_0}$$

Deep water depth,

$$h_0 = \frac{L_0}{2}$$

Wave number at a given water depth h where $h < h_0$,

$$k = \frac{h_0}{h} k_0$$

Table (1) presents the results of the δt calculation for wave period 8.0 sec and δx at water depth $h = 20.0$ m for some ε values.

Table (1) The results of δt and δx calculation

ε	δt (sec)	δx (m)
0.001	0.0025	0.0395
0.002	0.0051	0.079
0.003	0.0077	0.1186
0.004	0.0102	0.1583
0.005	0.0128	0.198
0.006	0.0153	0.2378
0.007	0.0179	0.2776
0.008	0.0205	0.3175

0.009	0.0231	0.3574
0.01	0.0256	0.3974

3.2. Results of the Airy Long-Wave Model.

Equations (4) and (5) were executed using a sinusoidal wave input:

$$\eta(0, t) = A \sin \sigma t$$

with a wave period of 8.0 seconds and a wave amplitude of 0.30 m at a water depth of 20.0 m. The model results are presented in Fig. (2) for one wave-period execution and in Fig. (3) for eight wave-period executions.

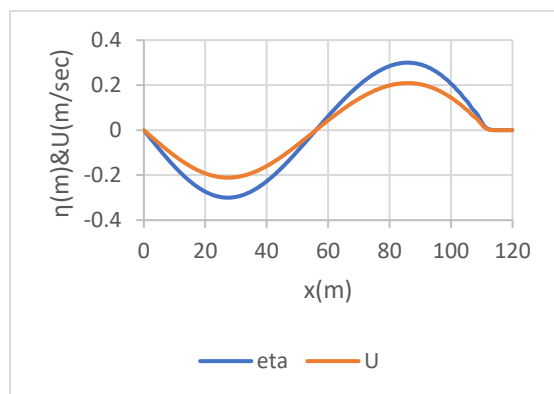


Fig.2: Wavelength model with input wave amplitude

$$A = 0.3 \text{ m}$$

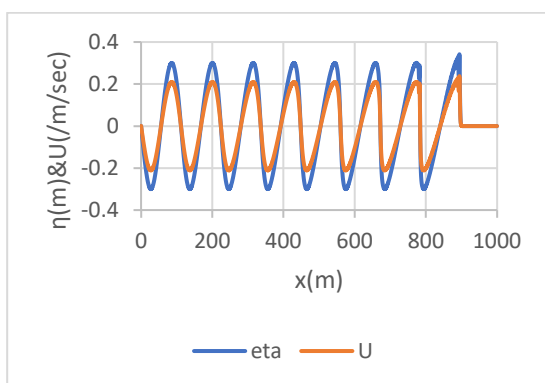


Fig.3: Model results with input wave amplitude

$$A = 0.3 \text{ m}$$

In Fig. (2), the wavelength is 110.0 m, whereas the wavelength predicted by linear wave theory is 88.793 m. Consequently, the estimated grid size based on the linear wave theory wavelength is smaller than the actual value, but it yields a more accurate representation. Furthermore, in the

eight-period execution, the wave profile becomes unstable at the wave crest near the boundary, indicating model failure. For a wave period of 8.0 seconds, the wave amplitude should ideally reach 1.20–1.30 m.

3.3. Convective Acceleration Assigned a Negative Sign.

There are two points that warrant attention in Equation (5). First, the velocity used is the depth-averaged velocity; therefore, the driving force on the right-hand side should be adjusted accordingly to match this velocity. Second, convective acceleration represents the difference in energy, which constitutes the hydrodynamic force. The direction of the positive hydrodynamic force is from a higher energy state toward a lower energy state, consistent with potential flow theory, in which positive velocity is defined as the flow from a higher velocity potential to a lower velocity potential:

$$u(x, z, t) = -\frac{\partial \phi}{\partial x} \quad \dots (10)$$

Where u is the water particle velocity in the horizontal x direction, and ϕ is the velocity-potential. Accordingly, the convective acceleration term should carry a negative sign. Thus, Equation (5) becomes:

$$\frac{\partial U}{\partial t} - \frac{1}{2} \frac{\partial U U}{\partial x} = -g \frac{\partial \eta}{\partial x} \quad \dots (11)$$

The results of the model execution using this equation, with the same water depth and wave period, and with a wave amplitude of 0.6 m, are presented in Fig. (4).

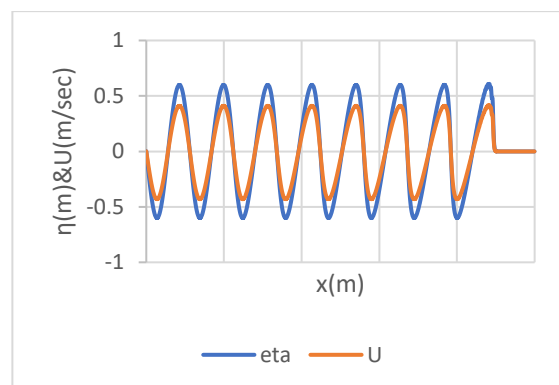


Fig.4: Model results with input wave amplitude

$$A = 0.6 \text{ m.}$$

As shown in Fig. (4), the model still produces stable results over eight wave periods. However, the model fails when the wave amplitude is increased to 0.70 m, for which the execution results are not presented.

3.4. Introducing a Coefficient to the Convective Acceleration Term.

Equation (5) was derived by integrating Equation (2) over the water depth. However, the integration method applied was not sufficiently accurate, as an integration coefficient of unity was used both in the continuity equation and in the momentum conservation equation. To reduce the error introduced by this approximation, the convective acceleration term is multiplied by a coefficient, such that Equation (5) becomes:

$$\frac{\partial U}{\partial t} - \frac{\mu}{2} \frac{\partial UU}{\partial x} = -g \frac{\partial \eta}{\partial x} \quad \dots (12)$$

At this stage, the coefficient μ was determined empirically through trial and error until the model was able to simulate a wave amplitude of 1.30 m at a wave period of 8.0 seconds.

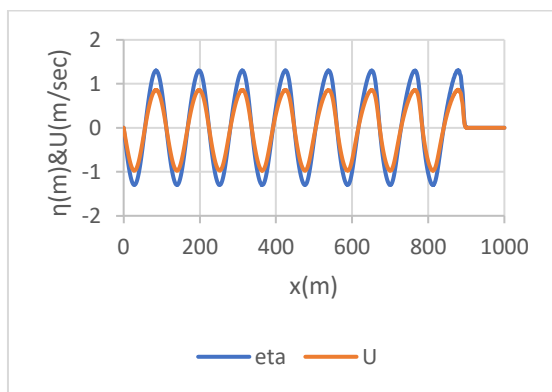


Fig.5: Model results with input wave amplitude

$A = 1.3 \text{ m}$.

Figure (5) presents the results of Equation (12) with a wave amplitude of 1.30 m, where a coefficient value of $\mu = 1.80$ was used. The model remained stable over eight wave periods.

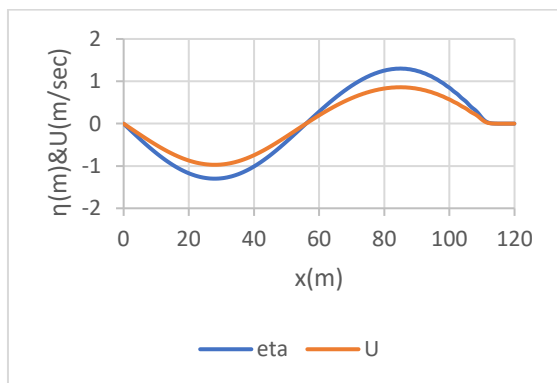


Fig.6: Wavelength model with $\mu = 1.8$, amplitude $A = 1.3 \text{ m}$.

As shown in the wavelength model in Fig. (6), the computed wavelength is 110.0 m, which shows no deviation from that obtained using the original formulation, Equation (5). At this stage, the coefficient μ is obtained empirically. To

determine its exact value, an alternative model is developed in the following section.

IV. ALTERNATIVE MODEL

In this section, an alternative model is formulated using a slightly different approach. Although the derivation method differs, the resulting equations are identical to those of the Airy long-wave model. In addition, this formulation provides exact coefficients for both the convective acceleration and the driving force terms.

4.1. Water surface elevation equation.

The water surface elevation equation is derived using the same method as in the Airy long-wave formulation, namely by integrating the continuity equation along the vertical z axis.

$$\int_h^\eta \frac{\partial u}{\partial x} dz + w_\eta - w_{-h} = 0$$

The first integral is evaluated using Leibniz's rule (Protter, Murray, Morrey, & Charles, 1985),

$$\int_{-h}^\eta \frac{\partial u}{\partial x} dz = \frac{\partial}{\partial x} \int_{-h}^\eta u dz - u_\eta \frac{\partial \eta}{\partial x} - u_{-h} \frac{\partial h}{\partial x}$$

Considering the kinematic seabed boundary condition,

$$u_{-h} \frac{\partial h}{\partial x} = -w_{-h}$$

$$\int_{-h}^\eta \frac{\partial u}{\partial x} dz = \frac{\partial}{\partial x} \int_{-h}^\eta u dz - u_\eta \frac{\partial \eta}{\partial x} + w_{-h}$$

Thus, the integrated continuity equation is expressed as,

$$\frac{\partial}{\partial x} \int_{-h}^\eta u dz - u_\eta \frac{\partial \eta}{\partial x} + w_{-h} + w_\eta - w_{-h} = 0$$

After combining similar terms, this integral becomes,

$$\frac{\partial}{\partial x} \int_{-h}^\eta u dz - u_\eta \frac{\partial \eta}{\partial x} + w_\eta = 0$$

The first integral is then evaluated using the concept of depth-averaged velocity:

$$\int_{-h}^\eta u dz = \beta_u UH$$

Where β_u is the integration coefficient, U is the depth-averaged horizontal velocity, and H is the total water depth. Substituting this expression and applying the kinematic free-surface boundary condition yields:

$$\frac{\partial \beta_u UH}{\partial x} - u_\eta \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} + u_\eta \frac{\partial \eta}{\partial x} = 0$$

After cancellation of similar terms, the water surface elevation equation becomes:

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \beta_u UH}{\partial x} \quad \dots (13)$$

If the integration coefficient is set to $\beta_u = 1$, this equation reduces to the same form as the water surface elevation equation (4).

4.2. Equation for Horizontal Water Particle Velocity.

The horizontal water particle velocity equation is derived using Equations (2) and (3), by first applying the property of irrotational flow: $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$, hence (2) and (3) become,

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (uu + ww) = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad \dots (14)$$

$$\frac{\partial w}{\partial t} + \frac{1}{2} \frac{\partial}{\partial z} (uu + ww) = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad \dots (15)$$

Equation (15) is expressed as an equation for pressure p which is then integrated with respect to the water depth from $z = z$ to $z = \eta$, while while applying the dynamic free-surface boundary condition $p_\eta = 0$, yielding the pressure distribution. Differentiating this pressure equation with respect to the horizontal axis- x and distributed it to (14),

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} \int_z^\eta \frac{\partial w}{\partial t} dz - \frac{1}{2} \frac{\partial}{\partial x} (u_\eta u_\eta + w_\eta w_\eta) - g \frac{\partial \eta}{\partial x}$$

Evaluating this equation at $z = \eta$, the surface velocity equation is obtained as:

$$\frac{\partial u_\eta}{\partial t} = -\frac{1}{2} \frac{\partial}{\partial x} (u_\eta u_\eta + w_\eta w_\eta) - g \frac{\partial \eta}{\partial x} \quad \dots (16)$$

This expression shows that both hydrodynamic and hydrostatic forces act as driving forces at the free surface.

The surface velocity is then converted into depth-averaged velocity using conversion coefficients,

$$u_\eta = \alpha_{u\eta} U$$

$$w_\eta = \alpha_{w\eta} W$$

$$\frac{\partial U}{\partial t} = -\frac{1}{2\alpha_{u\eta}} \frac{\partial}{\partial x} (\alpha_{u\eta}^2 UU + \alpha_{w\eta}^2 WW) - \frac{g}{\alpha_{u\eta}} \frac{\partial \eta}{\partial x} \quad \dots (17)$$

$\alpha_{u\eta}$ and $\alpha_{w\eta}$ are conversion coefficients relating surface velocity to depth-averaged velocity.

The Euler momentum conservation equation is then formulated under the same assumptions as in the continuity equation: horizontal velocity varies only along the horizontal axis, while vertical velocity varies only along the vertical axis. Furthermore, the hydrodynamic force is

assigned a negative sign, such that the positive direction corresponds to flow from higher kinetic energy toward lower kinetic energy. Accordingly, Equation (17) becomes:

$$\frac{\partial U}{\partial t} = \left(\frac{\alpha_{u\eta} \alpha_{u\eta}}{2} \frac{\partial UU}{\partial x} - g \frac{\partial \eta}{\partial x} \right) \frac{1}{\alpha_{u\eta}} \quad \dots (18)$$

This equation provides the velocity formulation, which has the same basic structure as the velocity equation in the Airy long-wave model. In this formulation, the driving force on the right-hand side is divided by the conversion coefficient, ensuring that the surface driving force is properly expressed in terms of the depth-averaged velocity. Consequently, the coefficient μ at (12) is $\alpha_{u\eta}$ or $\mu = \alpha_{u\eta}$.

4.3. Integration Coefficient and Conversion Coefficient.

The depth-averaged velocity is not the mean of all velocities at every elevation z across the entire water depth; rather, it corresponds to the velocity at a specific elevation $z = z_0$, Hutahaean (2024a, 2024b) as illustrated in Fig (10).

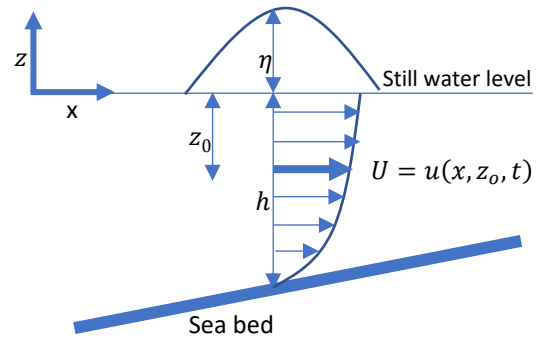


Fig. 7: Definition of depth-averaged velocity.

a. Integration coefficient β_u .

The depth-averaged velocity equation (Dean, 1991) is given by,

$$\int_{-h}^{\eta} u dz = \beta_u UH$$

which can be rearranged into the definition of the integration coefficient β_u ,

$$\beta_u = \frac{1}{UH} \int_{-h}^{\eta} u dz \quad \dots (19)$$

Using the velocity potential equation, the horizontal water particle velocity is expressed as:

$$u(x, z, t) = 2Gk \cos kx \cosh k(h + z) \sin \sigma t$$

Where G is the wave constant, k is the wave number, σ is the angular frequency.

By definition of the depth-averaged velocity:

$$U(x, t) = u(x, z_0, t)$$

$$= 2Gk \cos kx \cosh k(h + z_0) \sin \sigma t$$

For $z_0 = -\varepsilon h$,

$$U(x, t) = u(x, z_0, t)$$

$$= 2Gk \cos kx \cosh kh(1 - \varepsilon) \sin \sigma t$$

where $0 < \varepsilon < 1$. Substituting this velocity expression into Equation (15),

$$\beta_u = \frac{1}{2Gk \cos kx \cosh kh(1 - \varepsilon) \sin \sigma t H} \int_{-h}^{\eta} 2Gk \cos kx \cosh k(h + z_0) \sin \sigma t \, dz$$

After cancellation of common terms and completing the integration:

$$\beta_u = \frac{\sinh k(h + \eta)}{\cosh kh(1 - \varepsilon) kH}$$

According to the wave number conservation equation (Hutahaean, 2023),

$$k(h + \eta) = k \left(h + \frac{A}{2} \right) = \theta\pi$$

Where θ is the deep water coefficient, where $\tanh \theta\pi \approx 1$, thus

$$\beta_u = \frac{\sinh \theta\pi}{\theta\pi \cosh kh(1 - \varepsilon)}$$

The small amplitude approach is executed, thus $kh = \theta\pi$,

$$\beta_u = \frac{\sinh \theta\pi}{\theta\pi \cosh \theta\pi(1 - \varepsilon)} \quad \dots (20)$$

This equation enables calculation of the integration coefficient by determining the ε value.

b. Conversion Coefficient.

The relation between the surface horizontal velocity u_η and the depth-averaged horizontal velocity U is,

$$u_\eta = \alpha_{u\eta} U$$

Thus, the conversion coefficient $\alpha_{u\eta}$ is,

$$\alpha_{u\eta} = \frac{u_\eta}{U}$$

Substituting the velocity expressions yields:

$$\alpha_{u\eta} = \frac{2Gk \cos kx \cosh k(h + \eta) \sin \sigma t}{2Gk \cos kx \cosh kh(1 - \varepsilon) \sin \sigma t}$$

Obtaining,

$$\alpha_{u\eta} = \frac{\cosh \theta\pi}{\cosh \theta\pi(1 - \varepsilon)} \quad \dots (21)$$

Thus,

$$u_\eta u_\eta = \alpha_{u\eta}^2 U U \quad \dots (22)$$

As noted earlier, the deep-water coefficient θ is where $\tanh \theta\pi \approx 1$, which applies to very short-wavelength waves (Hutahaean, 2003). However, the velocity equation (18) yields wavelengths that are closer to those of linear wave theory, which are relatively long. Therefore, in this context the criterion is modified to $\tanh \theta\pi \rightarrow 1$. In this research, $\theta = 0.68$ and $\tanh \theta\pi = 0.972491$ is relatively close to 1. Examples of the integration coefficient β_u and the conversion coefficient $\alpha_{u\eta}$ for $\theta = 0.68$ is presented in Table (2).

Table (2) Integration coefficient and conversion coefficient

ε	β_u	$\alpha_{u\eta}$	$ \beta_u - 1 $
0.3960	0.999861	2.196404	0.000139
0.3961	1.000044	2.196807	0.000044
0.3962	1.000228	2.197211	0.000228
0.3963	1.000411	2.197614	0.000411
0.3964	1.000595	2.198017	0.000595
0.3965	1.000778	2.198421	0.000778
0.3966	1.000962	2.198824	0.000962
0.3967	1.001146	2.199228	0.001146
0.3968	1.001329	2.199631	0.001329
0.3969	1.001513	2.200035	0.001513
0.3970	1.001697	2.200438	0.001697

In this study, a value of $\beta_u \approx 1$ is used. As shown in Table (1), β_u closest to 1 is obtained from $\varepsilon = 0.3961$ with $\beta_u = 1.000044$ and $\alpha_{u\eta} = 2.196807$. The results of the model with an input wave amplitude of $A = 1.3 \text{ m}$ are shown in Fig.11.

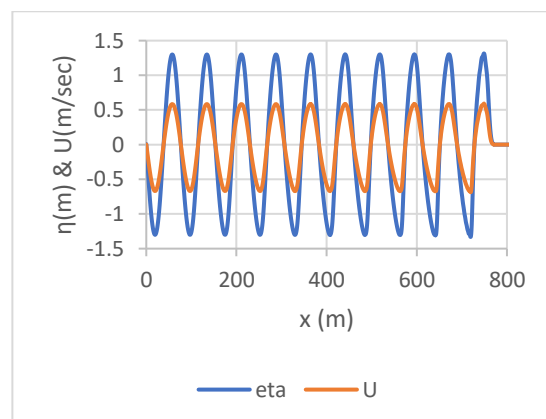


Fig.8: Alternative model results with an input wave amplitude $A = 1.3 \text{ m}$.

The execution of the alternative model in Fig. (8) demonstrates that the model remains stable after 10 wave

periods. Furthermore, two improvements are observed. First, the velocity is reduced, which indicates that this velocity corresponds to the flow velocity at a depth of z_0 below the surface. Second, the wavelength becomes shorter. As shown in Fig. (9), the wavelength produced by the alternative model is 75.0 m, which is shorter than the theoretical wavelength derived from linear wave theory, namely 88.793 m.

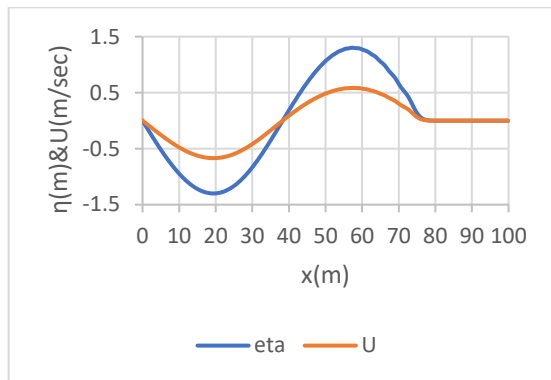


Fig (9). Wavelength of the alternative model, $T = 8$ sec.,
 $A = 1.3$ m, $h = 20.0$ m

From the analyses conducted, it can be concluded that the model can be improved by adjusting the sign of the convective acceleration term. It is also found that a conversion coefficient must be applied to the driving force when depth-averaged velocity is used as a variable in the model.

Despite the improvements made, the wavelength remains relatively long. Further refinement of the wavelength can be achieved by applying a weighted Taylor series to the Euler momentum conservation equation and the Kinematic Free Surface Boundary Condition (Hutahaeen, 2024a, 2024b).

It should be noted that this study does not aim to develop a time-series water wave model; rather, it focuses specifically on examining the convective acceleration term in the time-series water wave model.

V. CONCLUSION

The convective acceleration term in the time-series model represents the hydrodynamic force. The flow direction follows the gradient from higher energy to lower energy. Consequently, the positive direction of the hydrodynamic force is from higher kinetic energy to lower kinetic energy, consistent with velocity potential theory, in which flow moves from regions of higher potential velocity to lower potential velocity. Accordingly, the convective acceleration must be assigned a negative sign. This research also concludes that the application of depth-averaged velocity in

the velocity equation requires corresponding adjustments to the driving forces, including both hydrodynamic and hydrostatic components.

Furthermore, the acceleration equation in Euler's momentum conservation law is formulated through the Taylor series expansion, irrespective of the underlying physical phenomenon. This study demonstrates that each term in the Taylor series reflects a distinct physical process. Thus, the application of this series should be interpreted in relation to the physical phenomena it represents. In general, the use of mathematical equations in fluid dynamics must always be accompanied by consideration of the physical phenomena they are intended to model.

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